Annotations to a certain passage of Descartes for finding the quadrature of the circle*

Leonhard Euler

Summarium

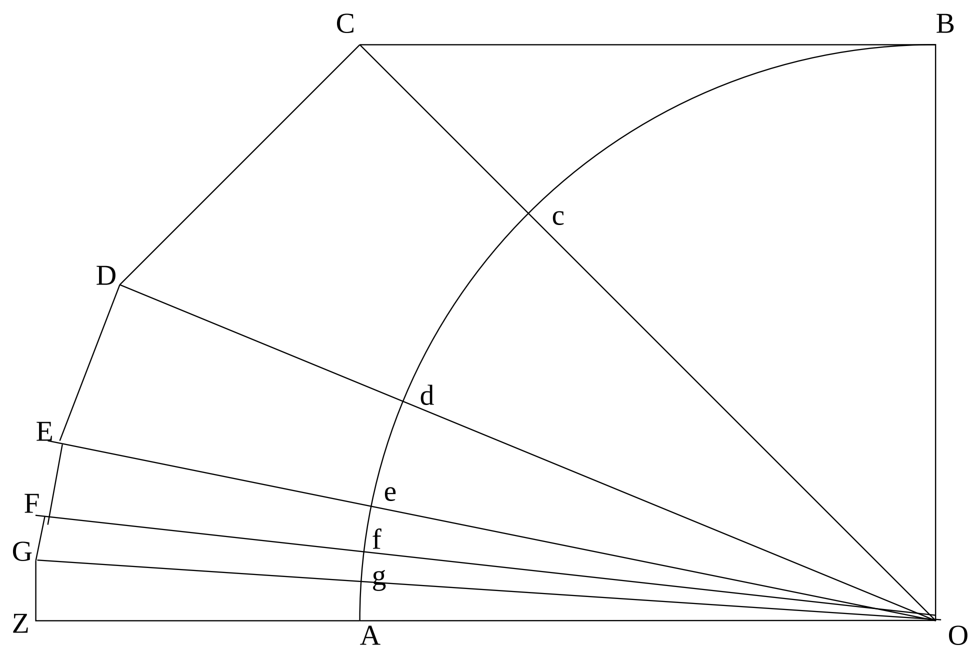
That the circumference of a circle is incommensurable with its diameter, or that no measure can be given that simultaneously measures both the diameter and the circumference of the circle,[1] has been observed already by the Geometers of antiquity, yet still now it cannot be demonstrated any further than that all attempts at finding a measure of this type have been in vain. Namely, no two numbers can be exhibited which hold between themselves the exact same ratio as occurs between the diameter and the circumference. Thus in practice one is wont to use numbers that closely approach this ratio, examples of which are 7 to 22 of Archimedes, and 113 to 355 of Metius; indeed this true ratio has been expressed then by others more accurately with numbers,[2] so that even in the computation of the largest circles the error is negligible. Knowing they are incommensurable though does not in itself prevent the ratio of the diameter to the circumference from being assigned geometrically, since the diagonal of a square is also incommensurable to the side, and in general all irrational quantities which arise from the extraction of roots can be constructed geometrically.[3] It seems that the true circumference of a circle belongs to a very high class of irrationalities, which can only be reached by repeating infinitely the process of extracting a root; here the best that can be done geometrically is to express more and more closely the true ratio of the circumference to the diameter. And the Cartesian construction that the Celebrated Author deals with here works in this way, such that by the continual apposition of rectangles which decrease according to a certain rule, a line is drawn which is finally equal to the circumference of a circle. This construction has been so ingeniously devised that by its simple use one is led quickly to the truth, and we should admire this extraordinary monument to the great insight of its discoverer. Euler, while

*Presented to the Berlin Academy on July 20, 1758. Originally published as *Annotationes in locum quendam Cartesii ad circuli quadraturam spectantem*, Novi Commentarii academiae scientiarum Petropolitanae **8** (1763), 157–168. Summarium is pp. 24–27. E275 in the Eneström index. Translated from the Latin by Jordan Bell, University of Toronto, Toronto, Ontario, Canada. Email: jordan.bell@gmail.com

[1]Translator: cf. book X, def. I.1 of Euclid.

[2]Translator: Here by “numbers” Euler means “integers”; cf. book VII, def. 2 of Euclid.

[3]Translator: I do not know if this means roots or just square roots. Indeed finding square roots can be done geometrically (with straightedge and compass), but finding the cube root cannot.



saving this discovery from oblivion, also publishes many singular formulas and series pertaining to measuring the circle, by which geometric approximations of this kind can be applied to a greater extent and further ones can be found. For instance, he has demonstrated that with q denoting the length of a quadrant of a circle whose radius is $= 1$,

$$q = \sec. \frac{1}{2}q \cdot \sec. \frac{1}{4}q \cdot \sec. \frac{1}{8}q \cdot \sec. \frac{1}{16}q \cdot \sec. \frac{1}{32}q \cdot \text{etc.},$$

from which one can conclude the following rather neat and elegant construction With the quadrant AOB established, the normal BC to the radius OB intersects the line OC bisecting the angle AOB at C. Then, the normal CD at C to this OC intersects the line OD bisecting the angle AOC at D. Similarly, DE, the line normal to this OD, intersects the line OE bisecting the angle AOD at E. Again then, EF, the line normal to OE, intersects the line OF bisecting the angle AOE at F, and so on. One continues in this manner until finally the radius OA is reached; this construction finally stops at the point Z. Then having done this, the line OZ will be precisely equal to the length of the quadrant $BcdefgA$.[4]

As well, one can easily derive many other constructions of this kind from the formulas of the Author. It will be helpful to note that the points B, C, D, E, F are found on a curve such that, by putting any angle $AOD = \phi$ and the line $OD = v$,

[4]Translator: Euler in fact does not explain why $OZ = BcdefgA$ later in the paper. The following explanation is from Ed Sandifer. OBC is a right angle so $\sec \frac{\pi}{4} = OC/OB$, hence $OC = \sec \frac{\pi}{4}$. As well, $\sec \frac{\pi}{8} = OD/OC$, hence $OD = \sec \frac{\pi}{4} \cdot \sec \frac{\pi}{8}$, etc. Then using the formula we get $OZ = BcdefgA$.

it turns out that $v = \frac{q \sin. \phi}{\phi}$.[5] Then, taking any ratio between the angle ϕ and the right angle, whose measure is the arc q, let $\phi = \frac{m}{n} q$; it will be $v = \frac{n}{m} \sin. \frac{m}{n} q$ and the line v can thus be assigned geometrically. Further, taking the angle ϕ as continually decreasing, it will come finally to a vanishing angle, for which $\frac{\sin. \phi}{\phi} = 1$, and then the line v is clearly equal to the quadrant q. One can make any number of additional similar formulas.

In Excerpts from the Manuscripts of *Descartes*,[6] a certain geometric construction which quickly approaches the true measure of the circle is briefly described. This construction, which either *Descartes* himself had found, or which had been communicated by someone else, especially at that time indicates brilliantly the insightful character of its discoverer. Those who later handled this same argument, as far as I know at least, have not made mention of this extraordinary construction, so that it is in danger that it disappear altogether into oblivion. This demonstration, which is given with nothing added to it, can be supplied without difficulty; truly not only the elegance of this fertile construction merits more study, but the notable conclusions that can be derived from it would altogether by themselves be worthy of attention. This most beautiful construction is proposed thus in the words of *Descartes* himself:

Fig. 1

> I find nothing more suitable for the quadrature of the circle than this: if to a given square bf is adjoined a rectangle cg contained by the lines ac and bc which is equal to a fourth part of the square bf: likewise a rectangle dh is made from the lines da, dc, equal to a fourth part of the preceding; and in the same way a rectangle ei, and further infinitely many others on to x: and this line ax will be the diameter of a circle whose circumference is equal to the perimeter of the square bf.

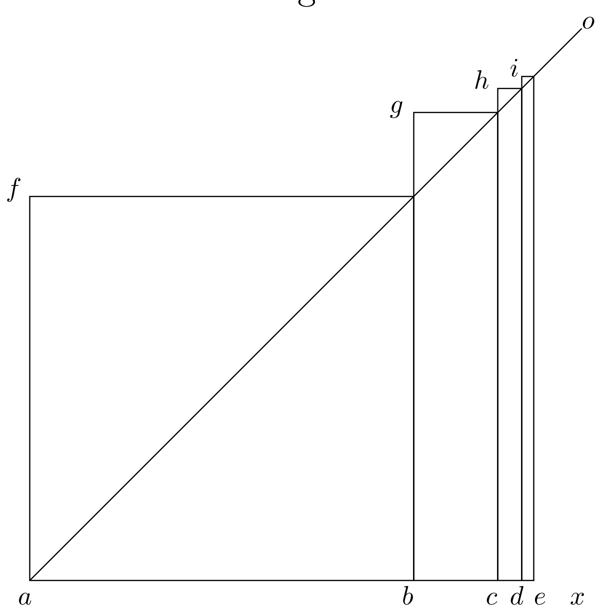
The strength of this method therefore consists in that by continually adjoining rectangles of this type, cg, dh, ei, etc., whose top right angles fall on the extended diagonal of the square, finally leading to the point x at which ends the diameter ax of a circle whose circumference is equal to the perimeter of the square bf, or four times the line ab.

Since each of these rectangles is equal to a fourth part of the preceding one, as is already observed by *Descartes* himself, it is clear that the sum of all these rectangles will be equal to a third part of the square bf; indeed this is clear

[5]Translator: This is the polar form of the quadratrix. Concerning finding points on the quadratrix see Christoph Clavius, *Commentaria in Euclidis Elementa*, appendix to book VI, pp. 296-304. Available online at http://mathematics.library.nd.edu/clavius/; also see pp. 317–318 of H. J. M. Bos, *On the representation of curves in Descartes' Géométrie*, Arch. Hist. Exact Sci. **24** (1981), no. 4, 295–338.

[6]Translator: See René Descartes, *R. Des-Cartes opuscula posthuma, physica, et mathematica*, 1701: part 6, Excerpta ex MSS. R. Des-Cartes, p. 6

Fig. 1



because the sum of the series

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \text{etc.}$$

continued to infinity is $= \frac{1}{3}$.

Descartes further indicates the rule on which this construction rests; he begins namely with regular polygons of $8, 16, 32, 64$, etc. sides, whose perimeters are mutually equal to the perimeter of the square bf. Now as ab is the diameter of a circle inscribed in this square, it is thus affirmed that ac is the diameter of a circle inscribed in an octagon, and indeed that ad and ae are the diameters of circles inscribed in a 16gon and 32gon respectively, and so on. Thus one sees that ax is the diameter of a circle inscribed in a polygon of infinitely many sides, whose circumference will therefore be equal to the perimeter of the square.

For clarity, I shall elaborate the demonstration of this construction. I observe that what are spoken of presently as the diameters of circles can equally well serve as radii, so that ab, ac, ad, ae, etc. can be seen to be radii of circles about which if regular polygons of $4, 8, 16, 32$, etc. sides are circumscribed, the perimeters of the polygons will be equal.

Problem

Given a circle about which a regular polygon has been circumscribed, to find another circle such that if a regular polygon with twice as many sides circumscribes it, the perimeter of the first polygon will be equal to the perimeter of the second polygon.

Solution

Fig. 2

Let ENM be the given circle, with center at C, and EP half of one side of the circumscribing polygon; further let CF be the radius of the circle that is being sought, and FQ half of one side of the polygon which is to circumscribe it. It is therefore necessary that FQ be half of EP and that the angle FCQ be half of the angle ECP. Let the line CQ bisect the angle ECP, and the line QO parallel to CE bisect the line EP. Now since[7]

$$\begin{aligned} & EV : CE = FQ : CF \\ \text{and} \quad & EV : CE = EP : CE + CP \\ \text{it will be} \quad & FQ : CF = EP : CE + CP \end{aligned}$$

but because $FQ = \frac{1}{2}EP$, it will further be

$$CF = \frac{1}{2}(CE + CP).$$

Then taking away CF one will have

$$EF = \frac{1}{2}(CP - CE)$$

from which the rectangle will be

$$CF \cdot EF = \frac{1}{4}(CP^2 - CE^2) = \frac{1}{4}EP^2,$$

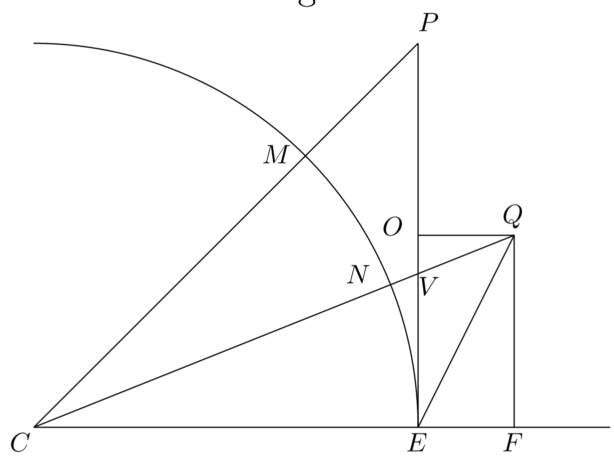
and so the point F should be defined such that the rectangle contained by CF and EF is equal to a fourth part of the square of the line EP, or to the square itself of the line FQ.

Corollary 1

Since $CF \cdot EF = FQ^2$, it will be $CF : FQ = FQ : EF$, whence having extended the line QE, the triangle FQE will be similar to the triangle FCQ, or ECV, and hence the angle FQE will be equal to the angle ECV.

[7] Translator: Ed Sandifer in his May 2008 *How Euler did it* works through the details of this paper. In particular, Sandifer explains that one can show $EV/CE = EP(CE+CP)$ using the identity $\cot\frac{\theta}{2} = \cot\theta + \sec\theta$; see pp. 151–152 of I. M. Gelfand and M. Saul, *Trigonometry* for this identity.

Fig. 2



Corollary 2

Since $CE : EV = EO : EF$, the point F may also be defined thus: from the point O the line normal to the line CV extended is drawn, and it will meet the base CE at F.

Corollary 3

If the polygon inscribed in the circle ENM has n sides, the angle ECP will be $= \frac{\pi}{n}$, where π denotes the measure of two right angles; and the angle $FCQ = \frac{\pi}{2n}$. Hence if the radius $CE = r$, it will be

$$EP = r \,\mathrm{tang.}\, \frac{\pi}{n} \quad \text{and} \quad FQ = \frac{1}{2} r \,\mathrm{tang.}\, \frac{\pi}{n}.$$

Corollary 4

Now because the angle $FQE = \frac{\pi}{2n}$, it will be

$$EF = FQ \,\mathrm{tang.}\, \frac{\pi}{2n} = \frac{1}{2} r \,\mathrm{tang.}\, \frac{\pi}{n} \,\mathrm{tang.}\, \frac{\pi}{2n}.$$

Indeed now, if we call $CF = s$, it will be

$$FQ = s \,\mathrm{tang.}\, \frac{\pi}{2n},$$

whence because

$$FQ = \frac{1}{2} r \operatorname{tang.} \frac{\pi}{n},$$

it will be

$$s = \frac{1}{2} r \operatorname{tang.} \frac{\pi}{n} \cot. \frac{\pi}{2n}.$$

Demonstration of Descartes' construction

Fig. 3

Here let CE be the radius of a circle inscribed in a square,[8] CF inscribed in an octagon, CG in a regular polygon of 16 sides, CH in a polygon of 32 sides and so on. Next let EP be half of one side of the square, FQ half of one side of the octagon, GR half of one side of the polygon with 16 sides, HS half of one side of the polygon with 32 sides, etc., and because these polygons are assumed to have the same perimeter, it will be

$$FQ = \frac{1}{2} EP, \quad GR = \frac{1}{2} FQ = \frac{1}{4} EP, \quad HS = \frac{1}{2} GR = \frac{1}{4} FQ = \frac{1}{8} EP, \quad \text{etc.}$$

Now from the previous problem, we have $CF \cdot EF = \frac{1}{4} EP^2 = FQ^2$; indeed, we get in the same way from it

$$CG \cdot FG = \frac{1}{4} FQ^2 = \frac{1}{4} CF \cdot EF = GR^2,$$
$$CH \cdot GH = \frac{1}{4} GR^2 = \frac{1}{4} CG \cdot FG = HS^2 \text{ etc.}$$

and the points F, G, H, etc. are plainly determined in the same way as was done in *Descartes'* construction; and because the intervals EF, FG, GH, etc. are made continually smaller, the final point x will be approached quickly enough. Then Cx will be the radius of the circle whose circumference is equal to the perimeter of the preceding polygons, and thus to eight times the line EP.
Q. E. D

Corollary 1

If one puts $CE = a$, $CF = b$, $CG = c$, $CH = d$, etc., the progression of these quantities is such that, as $EP = a$,

$$b(b-a) = \frac{1}{4} aa, \quad c(c-b) = \frac{1}{4} b(b-a), \quad d(d-c) = \frac{1}{4} c(c-b) \quad \text{etc.}$$

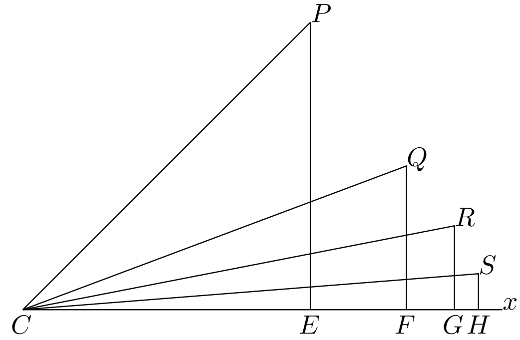
and hence

$$b = \frac{a + \sqrt{2aa}}{2}, \quad c = \frac{b + \sqrt{(2bb - ab)}}{2}, \quad d = \frac{c + \sqrt{(2cc - bc)}}{2} \quad \text{etc.}$$

and the limit of these quantities is the radius of a circle whose circumference is $= 8a$.

[8] Translator: Here we are inscribing the circle in n-gons which all have the same perimeter.

Fig. 3



Corollary 2

Since the angle ECP is half a right angle, or

$$ECP = \frac{\pi}{4},$$

the angles will be

$$FCQ = \frac{\pi}{8}, \quad GCR = \frac{\pi}{16}, \quad HCS = \frac{\pi}{32}, \quad \text{etc.}$$

Hence because

$$EP = a, \quad FQ = \frac{1}{2}a, \quad GR = \frac{1}{4}a, \quad HS = \frac{1}{8}a, \quad \text{etc.}$$

with cotangents it will be

$$CE = a\cot.\frac{\pi}{4}, \quad CF = \frac{1}{2}a\cot.\frac{\pi}{8}, \quad CG = \frac{1}{4}a\cot.\frac{\pi}{16}, \quad CH = \frac{1}{8}a\cot.\frac{\pi}{32} \quad \text{etc.}$$

Then with n denoting an infinite number, the last of these lines would

$$= \frac{1}{n}\cot.\frac{\pi}{4n}.$$

Corollary 3

But $\cot.\frac{\pi}{4n} = 1 : \text{tang.}\frac{\pi}{4n}$; and because the angle $\frac{\pi}{4n}$ is infinitely small, it will be

$$\text{tang.}\frac{\pi}{4n} = \frac{\pi}{4n} \quad \text{and hence} \quad \cot.\frac{\pi}{4n} = \frac{4n}{\pi}.$$

So were the last of these lines $= \frac{4a}{\pi}$, then the circumference of a circle described with this radius will be $= 2\pi \cdot \frac{4a}{\pi} = 8a$.

Corollary 4

Then, since by corollary 4 of the preceding problem[9]

$$EF = FQ\,\text{tang.}\,FCQ,$$

by the same rule it will be

$$FG = GR\,\text{tang.}\,GCR, \quad GH = HS\,\text{tang.}\,HCS \quad \text{etc.}$$

from which the other intervals can be expressed in the following way

$$EF = \frac{1}{2}a\,\text{tang.}\,\frac{\pi}{8}, \quad FG = \frac{1}{4}a\,\text{tang.}\,\frac{\pi}{16}, \quad GH = \frac{1}{8}a\,\text{tang.}\,\frac{\pi}{32} \quad \text{etc.},$$

with the first indeed in analogy

$$CE = a\,\text{tang.}\,\frac{\pi}{4} = a.$$

Corollary 5

With these combined with the preceding we will find

$$\begin{aligned}
CF &= a\Big(\text{tang.}\,\frac{\pi}{4} + \frac{1}{2}\,\text{tang.}\,\frac{\pi}{8}\Big) = \frac{1}{2}a\cot.\,\frac{\pi}{8},\\
CG &= a\Big(\text{tang.}\,\frac{\pi}{4} + \frac{1}{2}\,\text{tang.}\,\frac{\pi}{8} + \frac{1}{4}\,\text{tang.}\,\frac{\pi}{16}\Big) = \frac{1}{4}a\cot.\,\frac{\pi}{16},\\
CH &= a\Big(\text{tang.}\,\frac{\pi}{4} + \frac{1}{2}\,\text{tang.}\,\frac{\pi}{8} + \frac{1}{4}\,\text{tang.}\,\frac{\pi}{16} + \frac{1}{8}\,\text{tang.}\,\frac{\pi}{32}\Big) = \frac{1}{8}a\cot.\,\frac{\pi}{32}\\
&\text{etc.}
\end{aligned}$$

and so in this way, the sums of all these progressions can be assigned without any difficulty.

Corollary 6

Therefore progressing to infinity we will obtain the summation of the series

$$\text{tang.}\,\frac{\pi}{4} + \frac{1}{2}\,\text{tang.}\,\frac{\pi}{8} + \frac{1}{4}\,\text{tang.}\,\frac{\pi}{16} + \frac{1}{8}\,\text{tang.}\,\frac{\pi}{32} + \text{etc.} = \frac{4}{\pi},$$

which is thus obtained by the quadrature of the circle. I shall take occasion to solve the following problem.

[9]Translator: I don't see quite what corollary 4 of the preceding problem has to do with this. We know that $CF \cdot EF = FQ^2, CG \cdot FG = GR^2$, etc. from the Demonstration, from which $EF = FQ\,\text{tang.}\,FCQ, FG = GR\,\text{tang.}\,GCR$, etc. follow by similar triangles.

Problem

With ϕ denoting the arc of a circle whose radius is $= 1$, to find the sum of the infinite series

$$\text{tang.}\,\phi + \frac{1}{2}\,\text{tang.}\,\frac{1}{2}\phi + \frac{1}{4}\,\text{tang.}\,\frac{1}{4}\phi + \frac{1}{8}\,\text{tang.}\,\frac{1}{8}\phi + \frac{1}{16}\,\text{tang.}\,\frac{1}{16}\phi + \text{etc.}$$

Solution

If in Fig. 2, as constructed above, one sets the angle $ECP = \phi$, it will be $FCQ = \frac{1}{2}\phi$: now putting $FQ = 1$ it will be $EP = 2$, and hence

$$CE = 2\,\text{cot.}\,\phi, \quad CF = \text{cot.}\,\frac{1}{2}\phi \quad \text{and} \quad EF = \text{tang.}\,\frac{1}{2}\phi,$$

from which one has

$$2\,\text{cot.}\,\phi = \text{cot.}\,\frac{1}{2}\phi - \text{tang.}\,\frac{1}{2}\phi \quad \text{and} \quad \text{tang.}\,\frac{1}{2}\phi = \text{cot.}\,\frac{1}{2}\phi - 2\,\text{cot.}\,\phi.$$

and in the same way

$$\text{tang.}\,\phi = \text{cot.}\,\phi - 2\,\text{cot.}\,2\phi.$$

The values of the tangents found in the given series can be written by cotangents as

$$\begin{array}{rcl}
\text{tang.}\,\phi & = & \text{cot.}\,\phi - 2\,\text{cot.}\,2\phi, \\
\frac{1}{2}\,\text{tang.}\,\frac{1}{2}\phi & = & \frac{1}{2}\,\text{cot.}\,\frac{1}{2}\phi - \text{cot.}\,\phi, \\
\frac{1}{4}\,\text{tang.}\,\frac{1}{4}\phi & = & \frac{1}{4}\,\text{cot.}\,\frac{1}{4}\phi - \frac{1}{2}\,\text{cot.}\,\frac{1}{2}\phi, \\
\frac{1}{8}\,\text{tang.}\,\frac{1}{8}\phi & = & \frac{1}{8}\,\text{cot.}\,\frac{1}{8}\phi - \frac{1}{4}\,\text{cot.}\,\frac{1}{4}\phi \\
 & & \text{etc.}
\end{array}$$

and adding them together we will get

$$\begin{array}{r}
\text{tang.}\,\phi = \text{cot.}\,\phi - 2\,\text{cot.}\,2\phi, \\
\text{tang.}\,\phi + \frac{1}{2}\,\text{tang.}\,\frac{1}{2}\phi = \frac{1}{2}\,\text{cot.}\,\frac{1}{2}\phi - 2\,\text{cot.}\,2\phi, \\
\text{tang.}\,\phi + \frac{1}{2}\,\text{tang.}\,\frac{1}{2}\phi + \frac{1}{4}\,\text{tang.}\,\frac{1}{4}\phi = \frac{1}{4}\,\text{cot.}\,\frac{1}{4}\phi - 2\,\text{cot.}\,2\phi, \\
\text{tang.}\,\phi + \frac{1}{2}\,\text{tang.}\,\frac{1}{2}\phi + \frac{1}{4}\,\text{tang.}\,\frac{1}{4}\phi + \frac{1}{8}\,\text{tang.}\,\frac{1}{8}\phi = \frac{1}{8}\,\text{cot.}\,\frac{1}{8}\phi - 2\,\text{cot.}\,2\phi \\
\text{etc.,}
\end{array}$$

which continued to infinity will be $\frac{1}{n}\cot.\frac{1}{n}\phi = \frac{1}{\phi}$ if n denotes an infinity number, because tang. $\frac{1}{n}\phi = \frac{\phi}{n}$ and hence cot. $\frac{1}{n}\phi = \frac{n}{\phi}$. Hence the sum of the given series will be

$$\text{tang.}\,\phi + \frac{1}{2}\text{tang.}\,\frac{1}{2}\phi + \frac{1}{4}\text{tang.}\,\frac{1}{4}\phi + \frac{1}{8}\text{tang.}\,\frac{1}{8}\phi + \text{etc.} = \frac{1}{\phi} - 2\cot.2\phi.$$

Whence if 2ϕ is a right angle or $\phi = \frac{\pi}{4}$, then since cot. $\frac{\pi}{2} = 0$, the sum of the series would be $= \frac{1}{\phi} = \frac{4}{\pi}$, which was treated in the above case.

From this series many others can be derived which are no less noteworthy.

I. By differentiating this series we obtain

$$\frac{1}{\cos.\phi^2} + \frac{1}{4\cos.\frac{1}{2}\phi^2} + \frac{1}{4^2\cos.\frac{1}{4}\phi^2} + \frac{1}{4^3\cos.\frac{1}{8}\phi^2} + \frac{1}{4^4\cos.\frac{1}{16}\phi^2} + \text{etc.} = -\frac{1}{\phi\phi} + \frac{4}{\sin.2\phi^2};$$

or, since $\frac{1}{\cos.\phi} = \sec.\phi$ it will also be

$$(\sec.\phi)^2 + \frac{1}{4}\left(\sec.\frac{1}{2}\phi\right)^2 + \frac{1}{4^2}\left(\sec.\frac{1}{4}\phi\right)^2 + \frac{1}{4^3}\left(\sec.\frac{1}{8}\phi\right)^2 + \text{etc.} = \frac{1}{\sin.\phi^2\cos.\phi^2} - \frac{1}{\phi\phi}$$

II. Next, because $\cos.\phi^2 = \frac{1+\cos.2\phi}{2}$ and $\sin.2\phi^2 = \frac{1-\cos.4\phi}{2}$, by dividing everything by two it will be

$$\frac{1}{1+\cos.2\phi} + \frac{1}{4(1+\cos.\phi)} + \frac{1}{4^2(1+\cos.\frac{1}{2}\phi)} + \frac{1}{4^3(1+\cos.\frac{1}{4}\phi)} + \text{etc.}$$
$$= \frac{2}{1-\cos.4\phi} - \frac{1}{2\phi\phi}$$

or by writing $\frac{1}{2}\phi$ for ϕ

$$\frac{1}{1+\cos.\phi} + \frac{1}{4(1+\cos.\frac{1}{2}\phi)} + \frac{1}{4^2(1+\cos.\frac{1}{4}\phi)} + \frac{1}{4^3(1+\cos.\frac{1}{8}\phi)} + \text{etc.}$$
$$= \frac{2}{1-\cos.2\phi} - \frac{2}{\phi\phi}.$$

III. If the series which has been found is multiplied by $d\phi$ and integrated, because

$$\int d\phi\,\text{tang.}\,\phi = \int \frac{d\phi\sin.\phi}{\cos.\phi} = -l\cos.\phi \quad \text{and} \quad \int 2d\phi\cot.2\phi = l\sin.2\phi,$$

one will have

$$-l\cos.\phi - l\cos.\frac{1}{2}\phi - l\cos.\frac{1}{4}\phi - l\cos.\frac{1}{8}\phi - l\cos.\frac{1}{16}\phi - \text{etc.} = l\phi - l\sin.2\phi + \text{Const.}$$

In order to define this constant, let us put $\phi = 0$, and because $l\cos.0 = l1 = 0$ we will have 0 for the first part, while for the second part, since $\sin.2\phi = 2\phi$, we will have $l\phi - l2\phi + \text{Const.} = -l2 + \text{Const.}$, whence $\text{Const.} = l2$. Hence switching to numbers instead of the logarithms of numbers it will be

$$\frac{1}{\cos.\phi\cos.\frac{1}{2}\phi\cos.\frac{1}{4}\phi\cos.\frac{1}{8}\phi\cos.\frac{1}{16}\Phi\,\text{etc.}} = \frac{2\phi}{\sin.2\phi}$$

IV. Since

$$\frac{1}{\cos.\phi} = \sec.\phi,$$

this theorem can also be expressed according to secants as

$$\sec.\phi\sec.\frac{1}{2}\phi\sec.\frac{1}{4}\phi\sec.\frac{1}{8}\phi\sec.\frac{1}{16}\phi\,\text{etc.} = \frac{2\phi}{\sin.2\phi}.$$

From this, if the ratio of the diameter to the circumference is put $= 1 : \pi$ and q denotes a right angle, if we set $2\phi = q = \frac{\pi}{2}$ it will be

$$\sec.\frac{1}{2}q\sec.\frac{1}{4}q\sec.\frac{1}{8}q\sec.\frac{1}{16}q\sec.\frac{1}{32}q\,\text{etc.} = \frac{\pi}{2}.$$

Problem

To find a series of quantities: a, b, c, d, e, f, etc. which have the property that

$$c(c-b) = \frac{1}{4}b(b-a), \quad d(d-c) = \frac{1}{4}c(c-b), \quad e(e-d) = \frac{1}{4}d(d-c) \quad \text{etc.}$$

or such that the quantities thence formed

$$b(b-a), \quad c(c-b), \quad d(d-c), \quad e(e-d), \quad f(f-e), \quad \text{etc.}$$

decrease in quadruple ratio.

Solution

Since $\text{tang.}\frac{1}{2}\phi = \cot.\frac{1}{2}\phi - 2\cot.\phi$,[10] if we multiply each side by $\cot.\frac{1}{2}\phi$, since $\text{tang.}\frac{1}{2}\phi\cot.\frac{1}{2}\phi = 1$ it will be

$$\cot.\frac{1}{2}\phi\Big(\cot.\frac{1}{2}\phi - 2\cot.\phi\Big) = 1.$$

Let us then set

$$a = r\cot.\phi, \quad b = \frac{1}{2}r\cot.\frac{1}{2}\phi, \quad c = \frac{1}{4}r\cot.\frac{1}{4}\phi, \quad d = \frac{1}{8}r\cot.\frac{1}{8}\phi \quad \text{etc.},$$

[10] Translator: This follows from the addition formula for tan: $\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$.

and it will be

$$\frac{2b}{r}\left(\frac{2b}{r}-\frac{2a}{r}\right)=1, \quad \text{hence} \quad b(b-a)=\frac{rr}{4},$$
$$\frac{4c}{r}\left(\frac{4c}{r}-\frac{4b}{r}\right)=1, \quad \text{hence} \quad c(c-b)=\frac{rr}{4^2},$$
$$\frac{8d}{r}\left(\frac{8d}{r}-\frac{8c}{r}\right)=1, \quad \text{hence} \quad d(d-c)=\frac{rr}{r^3}$$
$$\text{etc.}$$

whence this series

$$a=r\cot\phi, \quad b=\frac{1}{2}r\cot.\frac{1}{2}\phi, \quad c=\frac{1}{4}r\cot.\frac{1}{4}\phi, \quad d=\frac{1}{8}r\cot.\frac{1}{8}\phi \quad \text{etc.}$$

has the property that the quantities thence formed

$$b(b-a), \quad c(c-b), \quad d(d-c), \quad e(e-d) \quad \text{etc.}$$

decrease in quadruple ratio.

Corollary 1

Given the first two terms a and b, all the remaining c, d, e, f are thence successively determined, such that

$$c=\frac{b+\sqrt{(2bb-ab)}}{2}, \quad d=\frac{c+\sqrt{(2cc-bc)}}{2}, \quad e=\frac{d+\sqrt{(2dd-cd)}}{2} \quad \text{etc.}$$

and hence with the first two terms taken at our pleasure, the entire series can be exhibited by means of these formulas.

Corollary 2

Moreover, with the terms a and b given, the angle ϕ and the quantity r can be thus defined from them

$$\text{tang.}\,\phi=\frac{2\sqrt{(bb-ab)}}{a} \quad \text{and} \quad r=2\sqrt{(bb-ab)};$$

then, having found the angle ϕ all the remaining terms can also be expressed, as

$$c=\frac{1}{4}r\cot.\frac{1}{4}\phi, \quad d=\frac{1}{8}r\cot.\frac{1}{8}\phi, \quad e=\frac{1}{16}r\cot.\frac{1}{16}\phi \quad \text{etc.}$$

Corollary 3

Hence the infinitesimal terms of this series will be $= \frac{r}{\phi}$, to which value the terms of the series converge fairly quickly.[11] Namely an arc is sought in the circle with radius $= 1$, whose tangent

$$= \frac{2\sqrt{(bb - ab)}}{a};$$

let this arc be $= \phi$, and the infinitesimal terms of our series will be

$$= \frac{2\sqrt{(bb - ab)}}{\phi}.$$

Scholion

Fig. 3

It will be useful to note here that the points P, Q, R, S, x (Fig. 3) are situated on the quadratrix of antiquity, because the line segments EP, FQ, GR, HS have the same ratio to each other as the angles ECP, FCQ, GCR, HCS, etc.[12] And since x, which is where this curve intersects the base, here as before has been found to indicate the quadrature of the circle, which is the very reason for the name of this curve, the construction of *Descartes* agrees singularly with this quadrature of antiquity, but it offers more conveniently and accurately the points E, F, G, H, etc. in succession, than what one could hope for by the continual bisection of angles.[13]

---

[11]Translator: $\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \cdots$, so the difference between any of c, d, e, etc. and $\frac{r}{\phi}$ is $O(\phi)$.

[12]Translator: Say $x_1 = y_1 \cot(\frac{y_1 \pi}{2})$, i.e. (x_1, y_1) is on the quadratrix. Using polar coordinates, $\frac{x_1}{y_1} = \cot \theta_1$. Suppose that $y_2 = \frac{y_1}{2}$ and $\theta_2 = \frac{\theta_1}{2}$. Writing (x_2, y_2) in polar coordinates we get $\frac{x_2}{y_2} = \cot \theta_2$. Then, in a few lines one can show using $x_1 = y_1 \cot(\frac{y_1 \pi}{2})$ and $\frac{x_1}{y_1} = \cot \theta_1$ that $x_2 = y_2 \cot(\frac{y_2 \pi}{2})$, i.e. that (x_2, y_2) is on the quadratrix.

[13]Translator: I don't see what Euler was thinking about when he says that Descartes' method is better than the one based on continued bisection of the angle. This statement needs clarification. References are given in the footnotes of the *Opera omnia*, I.15, pp. 1–15.